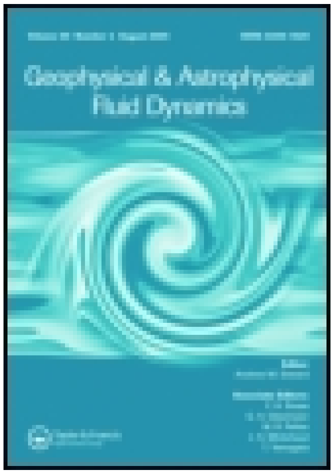

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On thermal convection in stratified fluids

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On Thermal Convection in Stratified Fluids

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The study of the mechanisms controlling the stratification in closed fluid regions is an important branch of geophysical fluid dynamics. Part of this subject can be handled with a simple linear model, consisting of a buoyancy layer at the non-horizontal boundaries of a container and an advective-diffusive interior coupled by volume continuity. The model is valid under the following conditions: firstly, the buoyancy-frequency characterizing the solution must be sufficiently large to give rise to a flow pattern of boundary layer type and, secondly, the non-horizontal walls must not have too large thermal conductivity.

The main purpose of the present paper is to summarise previous work done by the authors in this field and to present some consequences of their theory not previously discussed.

Three important cases are discussed; certain stationary solutions, the decay of a given stratification and the build up of a stratification in a homogeneous fluid. The experimental results concerning the afore-mentioned cases are presented.

1. INTRODUCTION

The problem of determining the interior stratification in closed or semi-closed containers has to some extent been investigated theoretically and experimentally by the authors. A theoretical study by Walin (1971, hereafter called "I") on the role of the buoyancy layer in a "strongly" stratified fluid subject to "weak" forcing at the non-horizontal boundaries showed that the stratification may be predicted for a fairly wide class of boundary conditions. A modified version of the theory was applied in laboratory experiments presented in a paper by Rahm and Walin (1978, hereafter called "II"). In a third paper by Rahm (1978, hereafter called "III") the main object was to study the transient behaviour of an initially homogeneous fluid. The purpose of the present paper is (i) to summarise the papers I–III, and (ii) to deal with some aspects of the theory not previously discussed explicitly.

The main features of the slightly extended theory presented in I are described in § 2. We are able to obtain a linear equation for the interior temperature field $T^I_{(z)}$ of a contained fluid. This equation is discussed in three steady-state cases. Some characteristic qualities of the stratification are shown in § 3.1, the effect of a net flux through the vessel is shown in § 3.2 and some consequences of the topography are discussed in § 3.3. The decay of a given stratification is discussed in § 4. The opposite is shown in the stratifying of an initially homogeneous fluid in § 5. It is believed that the mechanisms illustrated by these cases are important both in geophysical fluid dynamics and in industrial applications.

2. THEORETICAL BACKGROUND

2.1 Boundary conditions

For reasons to be discussed later, assume a closed or a semi-closed container of general shape defined by non-horizontal boundaries of finite thermal conductance. The container has a prescribed temperature $\hat{T}$ on the outside of the non-horizontal boundary. Its thickness is sufficiently small to allow the Newtonian boundary condition

$$\partial T/\partial\zeta = s_0(T-\hat{T}) \quad \text{at} \quad \zeta=0 \tag{2.1}$$

where s_0 is a prescribed function of position on the boundary. If the thermal conductivity of the wall and fluid are $\hat{k}$ and k respectively and the wall-thickness is d, then s_0 is defined by

$$s_0 = \hat{k}/kd$$

ζ measures the distance from the wall along the inward normal from the boundary. The velocity condition at the non-horizontal boundary is

$$\mathbf{V}=0 \tag{2.2}$$

A constant volume flux M_0 may be forced through the vessel from the bottom to the top, (see Fig. 1),

$$\iint_A w\,dA = M_0 \tag{2.3}$$

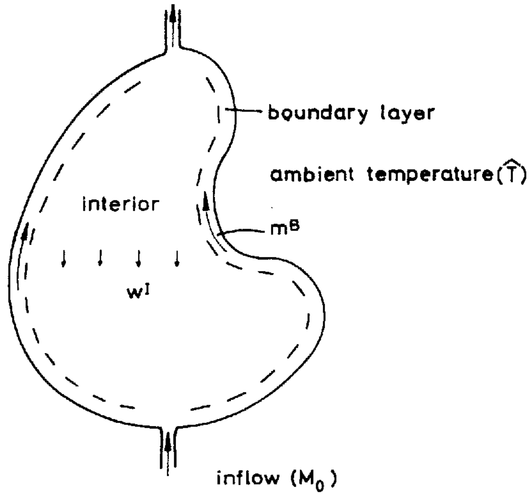

FIGURE 1 Schematic view of the almost-contained fluid. $T^I = T^I_{(z)}$ in the strongly stratified core. m^B is the buoyancy-layer transport and M_0 is the forced inflow at the bottom.

where w is the vertical velocity and A is the horizontal cross-section area of the region. The remaining boundary and initial conditions are presented in connection with specific cases.

2.2 The interior

The main features of the theory of the contained, stratified fluid are shown below. A careful analysis has been performed elsewhere by one of the authors (I). The total fields are built up as the sum of an interior and a boundary layer part given superscripts I and B respectively.

$$(\partial T^I/\partial x) \ll (\partial T^I/\partial z) \tag{2.4.a}$$

$$(\kappa/NL^2;\ \nu/NL^2) \ll 1 \tag{2.4.b}$$

$$s_0 L \ll (\nu/NL^2)^{-1/2} \tag{2.4.c}$$

where L is a typical length scale of the container, (κ, ν) are the diffusivities of heat and momentum and N is the buoyancy frequency. The theory is formulated within the Boussinesq approximation and condition (2.4.c) limits the heat flux through the non-horizontal boundary. The linearization of the interior equation is partly made possible by (2.4.a), leading to $T^I = T^I(z, t)$. The interior heat equation then becomes

$$\frac{\partial T^I}{\partial t} + W^I \frac{\partial T^I}{\partial z} = \kappa \frac{\partial^2 T^I}{\partial z^2} \tag{2.5}$$

and accordingly

$$W^I = W^I(z, t)$$

2.3 Boundary layer

The lowest order boundary layer equations (using the superscript B) are defined by the following equations:

$$u^B \frac{\partial T^I}{\partial \xi} = \kappa \frac{\partial^2 T^B}{\partial \zeta^2} \tag{2.6.a}$$

$$g\alpha T^B = \nu \frac{\partial^2 u^B}{\partial \zeta^2} \tag{2.6.b}$$

$$\frac{\partial u^B}{\partial \xi} + \frac{\partial w^B}{\partial \zeta} = 0 \tag{2.6.c}$$

These boundary-layer equations are derived under the assumption

$$T^B / T^I \ll 1 \tag{2.7}$$

(i.e. the heat flux through the non-horizontal wall is limited, (2.4.c)). Utilising (2.7) in the approximated boundary condition (2.1) the integration of (2.6.a) yields

$$m^B = \int_0^\infty u^B d\zeta = -\kappa \left\{ \frac{s_0(T^I - \hat{T}) - T^I_\zeta}{T^I_\xi} \right\}_{\zeta=0}$$

or

$$M^B = \oint m^B dl = \kappa \left\{ \frac{\partial A}{\partial z} - \oint s_0 \left\{ \frac{T^I - \hat{T}}{T^I_z} \right\}_{\zeta=0} \cos^{-1}\theta \, dl \right\} \tag{2.8}$$

The boundary layer transport consists of two parts, one driven by the heat flux through the boundary, the other caused by the slope of the wall.

M^B is expressed in the interior variable T^I only. Notice the singular behavior of m^B at the horizontal boundaries where another type of boundary layer develops (see (I)).

2.4 Governing equation

The integrated continuity equation becomes

$$M^B + W^I \cdot A(z) = M_0 \tag{2.9}$$

making use of (2.5) and (2.8) and (2.9) we obtain

$$\frac{\partial T^I}{\partial t} + \left\{ \frac{M_0}{A} - \frac{\kappa \partial A}{A \partial z} \right\} \frac{\partial T^I}{\partial z} + \frac{\kappa}{A} \left\{ \oint \frac{s_0 (T^I - \hat{T})}{\cos \theta} dl \right\}_{\zeta = 0} = \kappa \frac{\partial^2 T^I}{\partial z^2} \tag{2.10}$$

Note that the boundary condition at the non-horizontal wall is automatically satisfied by solutions to (2.10). Only the horizontal boundary conditions remain unsatisfied. If they are not horizontally dependent, (2.10) can itself satisfy these boundary conditions. The behaviour of this equation is discussed in the following paragraphs in some applications.

3. THREE STEADY CASES

We will now discuss three special cases of the theory given above for which solutions can be readily found. Each case illustrates one particular type of control of the stratification in the main body of the fluid. The buoyancy layer is of importance in all three cases. Vertical diffusion in the interior enters the basic heat balance in the first and the third case. Furthermore, in these cases a net vertical volume flux is imposed on the system.

3.1 Vertical temperature contrast

The most obvious way to establish a stratification in a fluid region is of course to keep its upper boundary at a higher temperature than its lower boundary. We will thus consider a cylindrical fluid region defined by

$$-H \leqq z \leqq H$$

$$r \leqq r_0$$

with the following boundary conditions

$$T=(T_1;\ T_2) \quad \text{at} \quad z=(-H;\ H) \tag{3.1a}$$

$$\frac{\partial T}{\partial r}=-s_0(T-\hat{T}) \quad \text{at} \quad r=r_0 \tag{3.1b}$$

We thus keep the upper horizontal boundary at $T=T_2$ and the lower at $T=T_1$, while the vertical boundary has a certain degree of insulation from the surroundings at temperature $T=\hat{T}$. Obviously if $s_0=0$, i.e. if the vertical boundary is completely insulated, the solution is simply a linear temperature distribution between the top and bottom of the region. We now ask how the system behaves if $s_0 \neq 0$. In particular we ask how small we have to make s_0 for the solution corresponding to $s_0=0$ to be approximately valid.

Applying the formalism of § 2 we find the solution

$$T^I=A_1 \exp -\alpha(z+H)+A_2 \exp \alpha(z-H)+\hat{T} \tag{3.2}$$

where $\alpha=(2s_0/r_0)^{1/2}$

$$A_1=\frac{(T_1-\hat{T})-(T_2-\hat{T})\exp(-2\alpha H)}{1-\exp(-4\alpha H)}$$

$$A_2=\frac{(T_2-\hat{T})-(T_1-\hat{T})\exp(-2\alpha H)}{1-\exp(-4\alpha H)}$$

The solution (3.2) and the associated flow pattern is illustrated in Fig. 2. We find that a linear temperature profile is created if

$$\alpha H \ll 1 \tag{3.3}$$

In the opposite case where

$$\alpha H \gg 1 \tag{3.4}$$

the temperature equals the surrounding temperature except in layers at the top and bottom of the system. Between these layers the top and bottom boundary conditions have no influence on the temperature distribution.

When the temperature profile is approximately linear the heat balance is essentially diffusive throughout the fluid. In the opposite case the

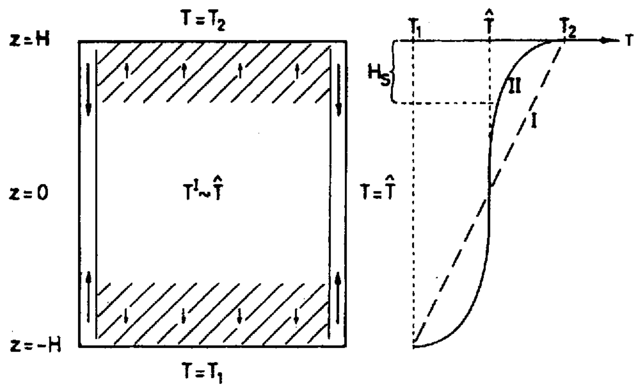

FIGURE 2 Flow pattern and temperature profile in fluid region with imposed temperature control on horizontal boundaries. Linear profile (I) results if sidewalls are extremely well insulated. Typical case (II) is characterized by penetration depth H_s which decreases when the insulation of the sides decreases.

control may be described as follows (see Fig. 2). In the upper (lower) part of the fluid where T^I deviates from $\hat{T}$ a downward (upward) transport occurs in the vertical boundary layer. Consequently an upward (downward) motion is created in the interior. The advective cooling (heating) associated with this vertical motion is balanced by vertical heat diffusion. The relative importance of diffusion increases when the upper (lower) scale height decreases. In fact the system will choose this scale height precisely so that the opposing effects, vertical diffusion and advection, will balance each other. The time required to establish the stationary solution is given by

(i) $\tau \sim H^2/\kappa$ where $\alpha H \ll 1$

(ii) $\tau \sim H_s^2/\kappa$ where $\alpha H \gg 1$

where we have introduced the penetration depth

$$H_s \sim \alpha^{-1} = \left(\frac{r_0}{2s_0}\right)^{1/2}$$

If, for example we were to establish a linear stratification (i.e. case (i)) in a 1 m deep container the time required would be of the order of one year.

3.2 An efficient stratification-control or a cylinder with vertical net flow

Let us again consider a cylindrical region but this time allow a net flow to pass vertically through the fluid region (II). We thus have a fluid inlet at $z = -H$ and an outlet at $z = H$. The temperature of the incoming fluid is controlled. As in the previous case we have

$$\partial T/\partial r = -s_0(T - \hat{T})$$

on the vertical boundary. The theoretical description applies only at some vertical distance from the in- and outlet where horizontal inhomogeneities evens out. Let us consider the effect when the net flux M_0 increases. For $M_0 = 0$ the system is described in § 3.1. When M_0 increases from zero the result is that the lower exponential is stretched out while the upper one is compressed. When

$$M_0 \gg (\kappa r_0^2)/H \tag{3.5}$$

the lower exponential dominates the temperature field except in a very thin upper layer which we neglect. Furthermore diffusion in the interior is unimportant everywhere except in this thin layer. In the regime defined by (3.5) the solution for the main part of the region is very well approximated by

$$T^I = \hat{T} + (T_0 - \hat{T}) \exp \beta(z + H) \tag{3.6}$$

where

$$\beta = \left(\kappa \oint s_0 dl \right) \Big/ M_0$$

and T_0 is the temperature at $z = -H$ (i.e. the temperature of the incoming fluid (see Fig. 3)). The physics of the system may be described as follows. The vertical transport through the system is carried out almost completely by the boundary layer transport. Thus at each level the boundary layer transport has to balance the imposed transport M_0, i.e.

$$M^B = M_0$$

This condition, together with the prescribed inlet temperature (which replaces a boundary condition on the upstream "wall"), completely

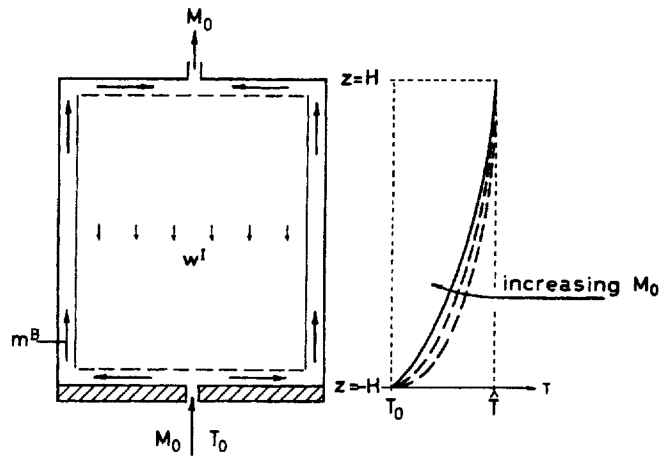

FIGURE 3 Circulation and temperature profile in a semi-closed cylinder when a net flow M_0 is forced through it with the temperature T_0 at the insulated bottom. The surroundings have the constant temperature $\hat{T}$. The main part of the interior transport takes place in the boundary layers. The effect of increasing M_0 upon the temperature profile is also shown.

determines the interior temperature field. (Equation (3.6) is more general than (3.5). In fact (3.6) may be applied to regions of quite general shape.) The described system (or modifications of it) is very efficient for the purpose of producing and maintaining any prescribed stratification in the laboratory. Comparing with the system described in § 3.1 we might say that we have turned the problem, (i.e. the incomplete insulation of the sidewall) into a tool to control the system. The time scale for the adjustment to steady-state is given by

$$\tau \sim H/(s_0 \kappa) \tag{3.8}$$

A typical value of s_0 may be $1\,\text{cm}^{-1}$ which gives $\tau = 1$ day with $H = 1$ m which is considerably smaller than the so-called diffusion time (i.e. (H^2/κ) ~ 1 year). Some experimental results are shown in Fig 4.

3.3 Influence of topography or an oversimplified estuary

The stratification in estuaries is dependent on the supply of dense water i.e. there is a net flow through the density structure. Furthermore the cross-sectional area always increases upwards (although more or less pronounced) and the sidewalls (i.e. the bottom) are essentially insulated.

Our third example (II) has the aforementioned properties. We thus consider a conical region (see Fig 5) defined by

$$A(z) = A_0(z/H)^2 \tag{3.9}$$

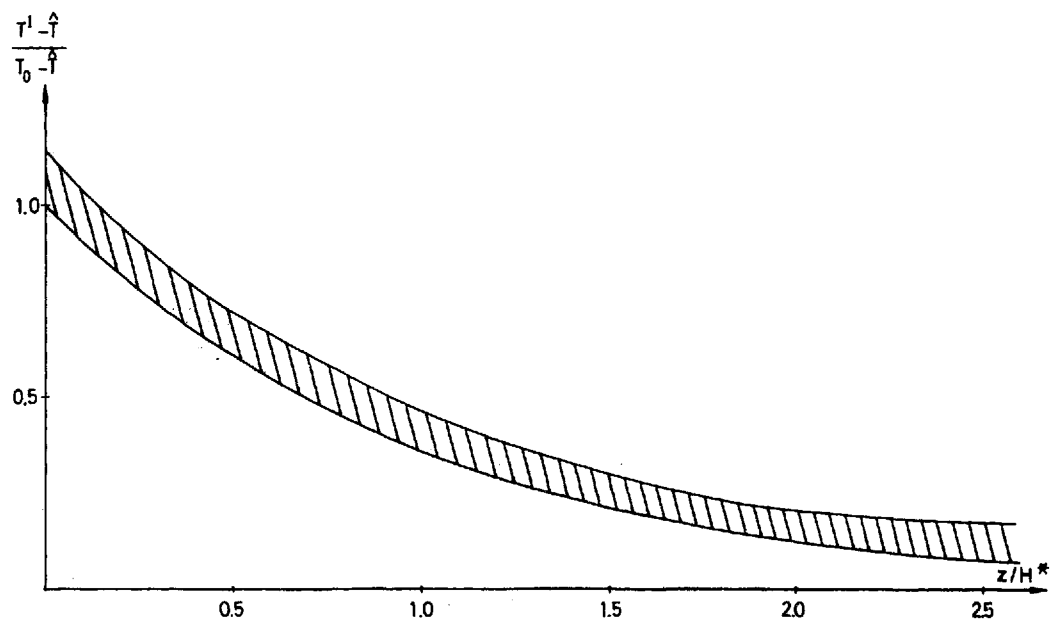

FIGURE 4 Temperature versus depth in non-dimensional form from experiments with a semi-closed cylinder. The radius of the cylinder was 14 cm and its height 33 cm. The "plexiglass" sidewall was 0.5 cm thick. The flow rate varied between 0.5 and 2.5 cm^3/s. The shadowed region contains 95% of all measured data. The lower solid line represents the theoretically predicted values.

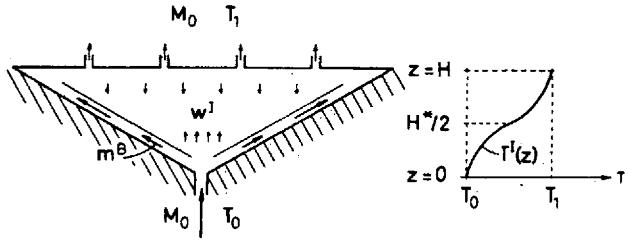

FIGURE 5 A net flux M_0 with the inflow temperature T_0 is forced through a cone with an insulated sidewall. The rigid top has the temperature T_1. A level of no vertical motion is shown in the interior. The temperature profile with the associated inflexion point is shown beside. The effect of increasing M_0 is to push this level upwards.

Furthermore, we specify a net flow M_0 upwards through the region. The fluid enters at the apex at $z=0$ with prescribed temperature T_0 (from a thermostated bath) and leaves the system somehow at $z=H$. The top plate is kept at the prescribed temperature T_1. Thus we have

$$T^I = T_1 \quad \text{at} \quad z=H$$

$$T^I = T_0 \quad \text{at} \quad z=0$$

The solution becomes

$$T^I = C_1 + C_2 \exp(-H^*/z) \tag{3.10}$$

where

$$C_1 = T_0$$

$$C_2 = (T_1 - T_0)\exp(-H^*/H)$$

$$H^* = \frac{M_0 H^2}{A_0 \kappa}$$

If the flow rate M_0 is not too large (i.e. $M_0 < 2(A_0\kappa)/H$) the stratification will show an inflexion point i.e. a level with maximal vertical density gradient. This level will move upwards if M_0 is increased. In order to understand the physics of the system we recall the fact that a sloping insulated boundary always produces a boundary layer flow. The net flow thus created may be expressed as

$$M^B = \kappa(\partial A/\partial z) \tag{3.11}$$

It is noteworthy that the magnitude of the flow is independent of the interior density gradient. We thus conclude that whatever the temperature profile looks like (as long as there is a stratification at all) a boundary layer flow given by (3.11) will occur. Given an imposed vertical flow M_0 we find that close to the apex $M_0 > M^B$. This means that $w^I > 0$ and that $(\partial T^I/\partial z^2) > 0$. w^I decreases upwards since an increasing part of the imposed flow M_0 is absorbed by M^B. Above the level $H^*/2$ (if $H^*/2 < H$) M^B will exceed M_0 and the vertical velocity w^I in the interior becomes negative and consequently $(\partial^2 T^I/\partial z^2) < 0$. Some experimental results are shown in Fig. 6.

4. DECAY OF A GIVEN STRATIFICATION

Let us consider a straight cylindrical container filled with an initially stratified fluid, $T^I_{init}(z)$. The horizontal boundaries are supposed to be insulated for simplicity. Now, what happens to the temperature field?

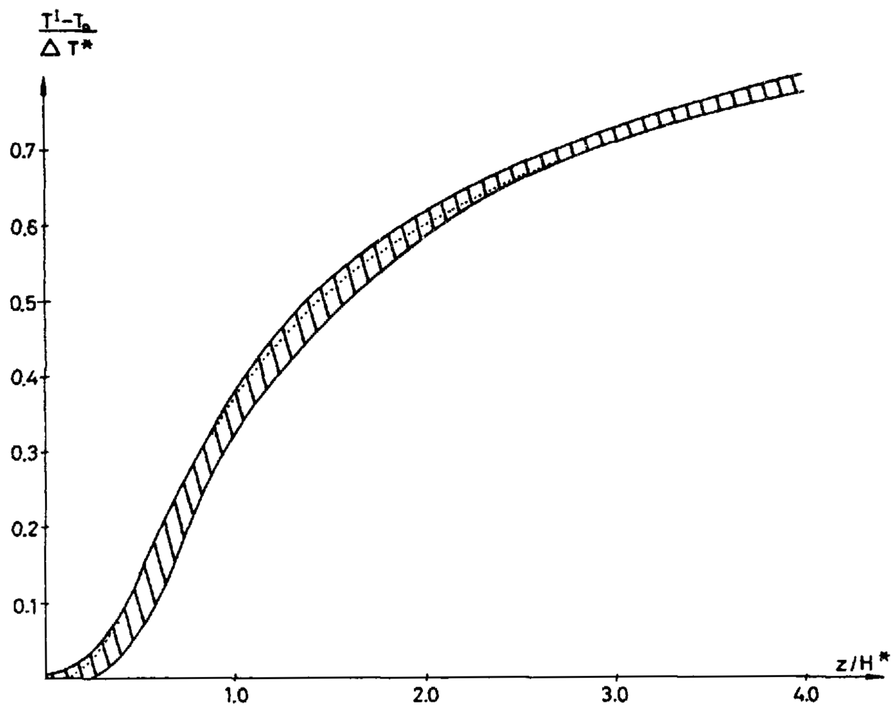

FIGURE 6 Temperature versus depth in non-dimensional form from experiments and theory with a conical vessel. The radius of the cone was 48 cm and its height was 6 cm. The flow rate M_0 varied between 0.2 and 1.5 cm^3/s. The shadowed area contains 95% of all measured data. The dotted line is the theoretically-predicted curve.

i) Pure diffusion will control the process if the sidewall is well insulated ($s_0 \sim 0$ and $\alpha H \ll 1$ as discussed in § 3.1). The buoyancy-layer is nonexistent to lowest order. The "diffusion time" is the relevant time scale for the decay.

ii) The effect of the diffusion is negligible if the thermal forcing at the sidewall is sufficiently large or according to § 3.1, $\alpha H \gg 1$. The governing equation (2.1) is reduced to

$$\frac{\partial T^I}{\partial t} + \frac{\kappa}{A_0} \oint s_0 (T^I - \hat{T}) dl = 0 \tag{4.1}$$

with the solution

$$T^I = T^I_{init}(z) \cdot \exp \left\{ - \frac{\left(\kappa \oint s_0 dl \right)}{A_0} \cdot t \right\} \tag{4.2}$$

The temperature decay is determined independently at each level irrespective of the stratification in the adjacent regions.

The magnitude of the difference $|T^I - \hat{T}|$ decreases with time with a constant rate and thus the shape of the initially given temperature profile is maintained (see Fig. 7). In thin horizontal boundary-layers of thickness H_s (§ 3.1) the solution is not valid. The advection is balanced by diffusion in these layers. The characteristic time scale for the decay is (if $\alpha H \gg 1$)

$$\tau \sim (H_s^2)/\kappa.$$

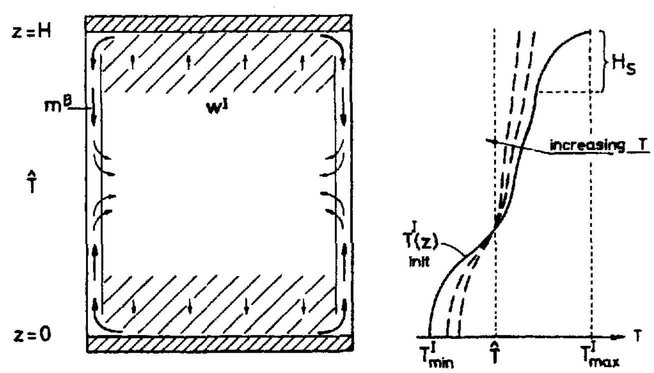

FIGURE 7 Flow pattern is shown for an initially-stratified fluid in a closed cylinder with insulated horizontal boundaries. The buoyancy layer transport m^B is divergent and alters the interior temperature at each level independently. The transient behavior is illustrated in the temperature profile. Note the thin boundary layers H_s at the horizontal walls.

5. RESPONSE TO AN AMBIENT TEMPERATURE CHANGE OR HOW TO STRATIFY AN HOMOGENEOUS FLUID BY MISTAKE

Again we use a cylindrical container with insulated horizontal boundaries that is filled with an initially homogeneous fluid of temperature $T^I_{init}=0$. The temperature outside the sidewall is $\hat{T}>0$ and we assume the thermal forcing to be sufficiently strong that $\alpha H \gg 1$. At $t=0$ a boundary-layer develops that transports slightly "warmer" fluid to the top of the vessel. This boundary-layer has been discussed by Goldstein (1938). A warmer region is thus formed above the homogeneous "colder" part of the interior. The "warmer" layer becomes stratified (as may be easily found by argument or by experimental observations (see Fig. 8)). As soon as a part

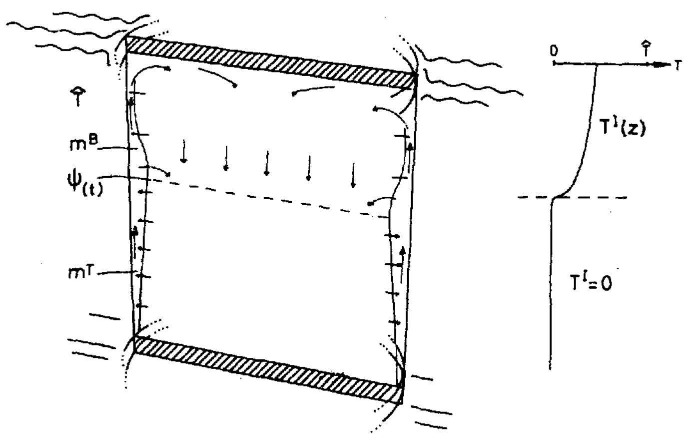

FIGURE 8 The initially homogeneous fluid is thermally forced at the vertical boundary. A stratified part is formed above the still homogeneous part of the interior. $\psi(t)$ shows the position of the front. The two types of boundary-layer transports is denoted m^T and m^B. The corresponding temperature profile is also shown.

of the interior has been stratified the adjacent boundary-layer changes into a buoyancy layer. The former boundary-layer transport m^T is used to determine the position of the "front", $\psi(t)$ that separates the homogeneous and stratified parts. Equation (4.1) is valid in the upper region and determines the decay of the stratification. This case has been studied by one of the authors (III).

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